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AN UPBOUND OF HAUSDORFF'S DIMENSION OF THE DIVERGENCE SET OF THE FRACTIONAL SCHRÖDINGER OPERATOR ON $H^s(\mathbb{R}^n)$*

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Abstract Given $n \geq 2$ and $\alpha > \frac{1}{2}$, we obtained an improved upbound of Hausdorff's dimension of the fractional Schrödinger operator; that is,

$$\sup_{f\in H^s(\mathbb{R}^n)} \dim_H \left\{x\in\mathbb{R}^n:\ \lim_{t\to 0} e^{\mathrm{i}t(-\Delta)^\alpha} f(x) \neq f(x)\right\} \leq n+1-\frac{2(n+1)s}{n}$$

for $\frac{n}{2(n+1)} < s \leq \frac{n}{2}$.

Key words The Carleson problem; divergence set; the fractional Schrödinger operator; Hausdorff dimension; Sobolev space

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1 Introduction

1.1 Statement of Theorem 1.1

Suppose that $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all functions $f:\mathbb{R}^n\to\mathbb{C}$ such that

$$f\in C^\infty(\mathbb{R}^n) \ \text{ and } \ \lim_{|x|\to\infty} x^\beta\partial^\gamma f(x)=0 \ \ \forall \ \text{ multi-indices } \ \beta,\gamma,$$

and let $H^s(\mathbb{R}^n)$ be the $\mathbb{R}\ni s$-Sobolev space of all tempered distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ whose Fourier transforms $\hat{f}$ obey

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} < \infty.$$

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If $(-\Delta)^\alpha f$ stands for the $(0,\infty)\ni\alpha$-pseudo-differential operator defined by the Fourier transformation acting on $f\in\mathcal{S}'(\mathbb{R}^n)$, that is, if

$$((-\Delta)^\alpha f)^\wedge(x)=|x|^{2\alpha}\hat{f}(x)\ \ \forall\ \ x\in\mathbb{R}^n,$$

then

$$u(x,t)=e^{\mathrm{i}t(-\Delta)^\alpha}f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}x\cdot\xi}e^{\mathrm{i}t|\xi|^{2\alpha}}\hat{f}(\xi)\mathrm{d}\xi \tag{1.1}$$

exists as a distributional solution to the α-Schrödinger equation

$$\begin{cases}\big(\mathrm{i}\partial_t+(-\Delta)^\alpha\big)u(x,t)=0\ \ \forall\ \ (x,t)\in\mathbb{R}^n\times\mathbb{R};\\ u(x,0)=f(x)\in H^s(\mathbb{R}^n).\end{cases} \tag{1.2}$$

Taking into account the Carleson problem of deciding such a critical regularity number s_c such that

$$\lim_{t\to0}e^{\mathrm{i}t(-\Delta)^\alpha}f(x)=f(x)\ \ \text{a.e.}\ \ x\in\mathbb{R}^n\ \ \text{holds for all}\ \ f\in H^s(\mathbb{R}^n)\ \ \&\ \ s>s_c, \tag{1.3}$$

we are trying to determine the Hausdorff dimension of the divergence set of the α-Schrödinger operator $e^{\mathrm{i}t(-\Delta)^\alpha}$, that is,

$$\mathrm{d}(s,n,\alpha)=\sup_{f\in H^s(\mathbb{R}^n)}\dim_H\left\{x\in\mathbb{R}^n:\lim_{t\to0}e^{\mathrm{i}t(-\Delta)^\alpha}f(x)\neq f(x)\right\}, \tag{1.4}$$

thereby discovering the case $\alpha>\frac{1}{2}$.

Theorem 1.1

$$\mathrm{d}(s,n,\alpha)\le n+1-\frac{2(n+1)s}{n}\ \ \text{under}\ \ n\ge2\ \ \&\ \ \alpha>\frac{1}{2}\ \ \&\ \ \frac{n}{2(n+1)}<s\le\frac{n}{2}. \tag{1.5}$$

1.2 Relevance of Theorem 1.1

Here, it is appropriate to say more about evaluating $\mathrm{d}(s,n,\alpha)$.

In general, we have the following development:

Theorem 1.1 actually recovers Cho-Ko's [1] a.e.-convergence result

$$f\in H^s(\mathbb{R}^n)\ \ \text{and}\ \ s>\frac{n}{2(n+1)}\ \ \&\ \ \alpha>\frac{1}{2}\Rightarrow\lim_{t\to0}e^{\mathrm{i}t(-\Delta)^\alpha}f(x)=f(x)\quad\text{a.e. } x\in\mathbb{R}^n.$$

A trivial part of Theorem 1.1 reveals that $\mathrm{d}(s,n<2s,\alpha)=0$ when $n\ge2\ \ \&\ \ \alpha>\frac{1}{2}$. Moreover, when $\alpha>\frac{1}{2}$, Theorem 1.1 improves (1.8) under

$$\frac{n}{2(n+1)}<s\le\frac{n+1}{4}$$

as follows:

in [2], Sjögren-Sjölin showed that

$$\mathrm{d}(s,n,\alpha)<n+1-2s\ \ \text{as}\ \ \frac{1}{2}<s\le\frac{n}{2}\ \ \&\ \ \alpha>\frac{1}{2}; \tag{1.6}$$

in [3] and [4], it was proved that

$$\mathrm{d}(s,n,\alpha)=n-2s\ \ \text{as}\ \ \frac{n}{4}\le s\le\frac{n}{2}\ \ \&\ \ \alpha=1. \tag{1.7}$$

when $\alpha \geq \frac{1}{2}$, Barceló-Bennett-Carbery et al. [3] showed that

$$\mathrm{d}(s,n,\alpha) \leq \begin{cases} n+1-2s & \text{as } \dfrac{1}{2} < s \leq \dfrac{n}{4}; \\ \dfrac{3n}{2}+1-4s & \text{as } \dfrac{n}{4} < s \leq \dfrac{n+1}{4}; \\ n-2s & \text{as } \dfrac{n+1}{4} < s \leq \dfrac{n}{2}. \end{cases} \tag{1.8}$$

In particular, we have the following case-by-case treatment:

Case $\alpha = 1$. Under this setting, Theorem 1.1 coincides with Du-Zhang's [5, Theorem 2.4], since (1.1) turns out to be the classical Schödinger operator $e^{-it\Delta}$. (1.3) was first proposed in [6] by Carleson for this special case, and then intensively studied in [7–17]. Upon combining the results in [5, 6, 9, 18, 19], we conclude that $s_c = \frac{n}{2(n+1)}$. Furthermore, in [2], Sjögren-Sjölin considered $\mathrm{d}(s,n,1)$. Note that the Sobolev embedding ensures that $\mathrm{d}(s,n<2s,1)=0$, so it is enough to calculate $\mathrm{d}(s,n\geq 2s,1)$.

Bourgain's counterexample in [9] and Lucà-Rogers' result in [20] showed that

$$\mathrm{d}(s,n,1) = n \quad \text{as} \quad s \leq \frac{n}{2(n+1)}.$$

The results in Žubrinić [4] and Barceló-Bennett-Carbery et al. [3] found that

$$\mathrm{d}(s,n,1) = n-2s \quad \text{as} \quad \frac{n}{4} \leq s \leq \frac{n}{2}.$$

Accordingly,

$$\frac{n}{2(n+1)} = \frac{n}{4} = \frac{1}{4} \Rightarrow \mathrm{d}(s,1,1) = 1-2s.$$

On the one hand, in [5], Du-Zhang proved that

$$\mathrm{d}(s,n,1) \leq n+1-\frac{2(n+1)s}{n} \quad \text{as} \quad \frac{n}{2(n+1)} < s < \frac{n}{4} \quad \& \quad n \geq 2.$$

On the other hand, in [20, 21], Lucà-Rogers obtained that

$$\mathrm{d}(s,n,1) \geq \left\{ \begin{cases} n+\dfrac{n}{n-1}-\dfrac{2(n+1)s}{n-1} & \text{as } \dfrac{n}{2(n+1)} \leq s < \dfrac{n+1}{8}; \\ n+1-\dfrac{2(n+2)s}{n} & \text{as } \dfrac{n+1}{8} \leq s < \dfrac{n}{4}. \end{cases} \right.$$

Thus there is still a gap in terms of determining the exact value of $\mathrm{d}(s,n,1)$; see also [5, 20–23] for more information.

Case $\alpha \in (\frac{1}{2},\infty)$. Sjölin [13] proved that $s_c = \frac{1}{2^2}$ for $n=1$. By the iterative argument developed in [8], Miao-Yang-Zheng [11] proved that (1.3) holds for

$$s > \frac{3}{8} \quad \text{and} \quad n = 2.$$

Very recently, Cho-Ko [1] proved that (1.3) holds for

$$s > \frac{n}{2(n+1)} \quad \text{and} \quad n \geq 2.$$

It seems that the case $\alpha > \frac{1}{2}$ shares the same critical index with the case $\alpha = 1$. So far there has been no counterexample to verify this problem.

Case $\alpha \in (0, \frac{1}{2}]$. It is uncertain that Theorem 1.1 can be extended to the fractional Schrödinger operator $e^{it(-\Delta)^\alpha}$ and $0 < \alpha \le \frac{1}{2}$; an investigation of this extension, coupled with the foregoing counterexample, will be the subject of future articles.

Throughout the rest of this paper, we always assume that $\alpha > \frac{1}{2}$.

In Section 2, we verify Theorem 1.1 by Proposition 2.1 and Theorem 2.2, which provides global L^1 and local L^2 estimates for the maximal operator living on a compactly-supported Borel measure and $e^{\mathrm{i}t(-\Delta)^\alpha}f(x)$. The proof of Theorem 2.2 is given in Section 3 by Theorem 3.1, which gives an $L^{\frac{2(n+1)}{n-1}}$ estimate for $e^{\mathrm{i}t(-\Delta)^\alpha}f(x)$, and Corollary 3.2, which gives an L^2-estimate for $e^{\mathrm{i}t(-\Delta)^\alpha}f(x)$. Thanks to a nontrivial analysis, Section 4 is devoted to presenting a proof of Theorem 3.1 which essentially relies on Theorems 4.1 and 4.4 which give the broad $1 \le k \le n+1$ linear refined Strichartz estimates in dimension $n+1$, and Lemma 4.5 which provides the narrow $L^{\frac{2(n+1)}{n-1}}$ estimate for $e^{\mathrm{i}t(-\Delta)^\alpha}f(x)$.

Notation In what follows, $A \lesssim B$ stands for $A \le CB$ for a constant $C > 0$, and $A \sim B$ means that $A \lesssim B \lesssim A$. Furthermore, for a given large number R and small enough $0 < \epsilon < 1$, $A \lessapprox B$ stands for $A \le CR^\epsilon B$ for a constant $C > 0$, and $A \approx B$ means that $A \lessapprox B \lessapprox A$.

2 Theorem 2.2⇒ Theorem 1.1

2.1 Proposition 2.1 and its Proof

In order to determine the Hausdorff dimension of the divergence set of $e^{\mathrm{i}t(-\Delta)^\alpha}f(x)$, we need a law for $H^s(\mathbb{R}^n)$ to be embedded into $L^1(\mu)$ with a lower dimensional Borel measure μ on $\mathbb{R}^n$.

Proposition 2.1 For a nonnegative Borel measure μ on $\mathbb{R}^n$ and $0 \le \kappa \le n$, let

$$C_\kappa(\mu) = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\kappa}\mu\big(B^n(x,r)\big) \quad \text{with} \quad B^n(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\},$$

and let $M^\kappa(\mathbb{B}^n)$ be the class of all probability measures μ with $C_\kappa(\mu) < \infty$ that are supported in the unit ball $\mathbb{B}^n = B^n(0,1)$. Suppose that

$$\begin{cases} 0 < s \le \dfrac{n}{2}; \\ \kappa > \kappa_0 \ge n - 2s; \\ (N, f, \mu) \in [1,\infty) \times H^s(\mathbb{R}^n) \times M^\kappa(\mathbb{B}^n); \\ \psi(r) = \exp(-r^2); \\ e_N^{\mathrm{i}t(-\Delta)^\alpha} f(x) = \dfrac{1}{(2\pi)^n}\displaystyle\int_{\mathbb{R}^n} \psi\left(\frac{|\xi|}{N}\right) e^{\mathrm{i}(x\cdot\xi + t|\xi|^{2\alpha})}\hat{f}(\xi)\mathrm{d}\xi. \end{cases}$$

(i) If $t \in \mathbb{R}$, then

$$\left\|\sup_{1\le N<\infty}\left|e_N^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))} \lesssim \sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)}. \tag{2.1}$$

(ii) If

$$\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))} \lesssim \sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)}, \tag{2.2}$$

then $\mathrm{d}(s,n,\alpha) \le \kappa_0$.

Proof (i) (2.1) is the elementary stopping-time-maximal inequality [3, (4)].

(ii) The argument is split into two steps.

Step 1 We show the following inequality:

$$\left\|\sup_{0<t<1}\sup_{N\geq 1}\left|e_N^{\mathrm{i}t(-\Delta)^\alpha}f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\lesssim\sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)}. \tag{2.3}$$

In a similar way as to the verification of [3, Proposition 3.2], we achieve

$$\sup_{N\geq 1}\left|e_N^{\mathrm{i}t(-\Delta)^\alpha}f(x)\right|\leq\left|e_1^{\mathrm{i}t(-\Delta)^\alpha}f(x)\right|+\int_1^\infty\left|\frac{\mathrm{d}}{\mathrm{d}N}e_N^{\mathrm{i}t(-\Delta)^\alpha}f(x)\right|\mathrm{d}N.$$

It is not hard to obtain (2.3) if we have the inequalities

$$\left\|\sup_{0<t<1}\left|e_1^{\mathrm{i}t(-\Delta)^\alpha}f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\lesssim\sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)} \tag{2.4}$$

and

$$\int_1^\infty\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha}\left(\frac{(\cdot)}{N^2}\psi'\left(\frac{(\cdot)}{N}\right)\hat{f}(\cdot)\right)^\vee\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\mathrm{d}N\lesssim\sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)}. \tag{2.5}$$

(2.4) follows from the fact that (2.2) implies

$$\begin{aligned}\left\|\sup_{0<t<1}\left|e_1^{\mathrm{i}t(-\Delta)^\alpha}f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}&=\left\|\sup_{0<t<1}\left|\int_{\mathbb{R}^n}e^{\mathrm{i}(x\cdot\xi+t|\xi|^{2\alpha})}\psi(\xi)\hat{f}(\xi)\mathrm{d}\xi\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\\&=\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha}\left(\psi(\cdot)\hat{f}(\cdot)\right)^\vee\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\\&\lesssim\sqrt{C_\kappa(\mu)}\left\|\left(\psi(\cdot)\hat{f}(\cdot)\right)^\vee\right\|_{H^s(\mathbb{R}^n)}\\&\lesssim\sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

To prove (2.5), we utilize

$$\psi'\left(\frac{|\xi|}{N}\right)\lesssim\sum_{k\geq 0}2^{-2nk}\chi_{B^n(0,2^kN)}(\xi)$$

to calculate

$$\begin{aligned}\left\|\left(\frac{\psi'\left(\frac{(\cdot)}{N}\right)(\cdot)\hat{f}(\cdot)}{N^2}\right)^\vee\right\|_{H^s(\mathbb{R}^n)}&\lesssim\left\|\frac{(1+|\cdot|^2)^{\frac{s}{2}}\sum\limits_{k\geq 0}2^{-2nk}\chi_{B^n(0,2^kN)}(\cdot)(\cdot)\hat{f}(\cdot)}{N^2}\right\|_{L^2(\mathbb{R}^n)}\\&\leq\sum_{k\geq 0}\frac{2^{-2nk}}{N^{1+\epsilon}}\left\|\frac{(1+|\cdot|^2)^{\frac{s}{2}}\chi_{B^n(0,2^kN)}(\cdot)(\cdot)\hat{f}(\cdot)}{N^{1-\epsilon}}\right\|_{L^2(\mathbb{R}^n)}\\&\lesssim\frac{1}{N^{1+\epsilon}}\|f\|_{H^{s+\epsilon}(\mathbb{R}^n)}.\end{aligned} \tag{2.6}$$

By (2.2) and (2.6), we obtain

$$\int_1^\infty\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha}\left(\frac{\psi'\left(\frac{(\cdot)}{N}\right)(\cdot)\hat{f}(\cdot)}{N^2}\right)^\vee\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\mathrm{d}N$$

$$\begin{aligned}
&\lesssim \int_1^\infty \sqrt{C_\kappa(\mu)} \left\| \left(\frac{\psi'\left(\frac{(\cdot)}{N}\right)(\cdot)\hat{f}(\cdot)}{N^2} \right)^\vee \right\|_{H^s(\mathbb{R}^n)} \mathrm{d}N \\
&\lesssim \int_1^\infty \sqrt{C_\kappa(\mu)} \frac{1}{N^{1+\epsilon}} \|f\|_{H^{s+\epsilon}(\mathbb{R}^n)} \mathrm{d}N \\
&\lesssim \sqrt{C_\kappa(\mu)} \|f\|_{H^{s+\epsilon}(\mathbb{R}^n)},
\end{aligned}$$

thereby reaching (2.5).

Step 2 We now show that

$$\mathrm{d}(s,n,\alpha) \le \kappa_0 \ \ \forall \ \ \kappa_0 \in [n-2s, \kappa).$$

By the definition, we have

$$\begin{aligned}
&\mu \left\{ x \in \mathbb{B}^n : \lim_{t\to 0} e^{\mathrm{i}t(-\Delta)^\alpha} f(x) \ne f(x) \right\} \\
=& \mu \left\{ x \in \mathbb{B}^n : \lim_{t\to 0} \lim_{N\to\infty} e_N^{\mathrm{i}t(-\Delta)^\alpha} f(x) \ne \lim_{N\to\infty} e_N^{\mathrm{i}0(-\Delta)^\alpha} f(x) \right\}.
\end{aligned} \tag{2.7}$$

For any

$$f \in H^s(\mathbb{R}^n) \ \text{ and } \ 0 < \epsilon \ll 1$$

there exists

$$g \in \mathcal{S}(\mathbb{R}^n) \ \text{ such that } \ \|f-g\|_{H^s(\mathbb{R}^n)} < \epsilon.$$

Accordingly, if

$$\mu \in M^\kappa(\mathbb{B}^n) \text{ and } \kappa > \kappa_0 \ge n-2s,$$

then a combination of (2.3) and (2.1) gives that

$$\begin{aligned}
&\mu \left\{ x \in \mathbb{B}^n : \overline{\lim_{t\to 0}}\, \overline{\lim_{N\to\infty}} \left| e_N^{\mathrm{i}t(-\Delta)^\alpha} f(x) - e_N^{\mathrm{i}0(-\Delta)^\alpha} f(x) \right| > \lambda \right\} \\
\le& \mu \left\{ x \in \mathbb{B}^n : \sup_{0<t<1} \sup_{N\ge 1} \left| e_N^{\mathrm{i}t(-\Delta)^\alpha} (f-g)(x) \right| > \frac{\lambda}{3} \right\} \\
&+ \mu \left\{ x \in \mathbb{B}^n : \lim_{t\to 0} \lim_{N\to\infty} \left| e_N^{\mathrm{i}t(-\Delta)^\alpha} g(x) - e_N^{\mathrm{i}0(-\Delta)^\alpha} g(x) \right| > \frac{\lambda}{3} \right\} \\
&+ \mu \left\{ x \in \mathbb{B}^n : \sup_{N\ge 1} \left| e_N^{\mathrm{i}0(-\Delta)^\alpha} (g-f)(x) \right| > \frac{\lambda}{3} \right\} \\
\le& \lambda^{-1} \left\| \sup_{0<t<1} \sup_{N\ge 1} \left| e_N^{\mathrm{i}t(-\Delta)^\alpha} (f-g) \right| \right\|_{L^1(\mathbb{B}^n; \mathrm{d}\mu(x))} \\
&+ \lambda^{-1} \left\| \sup_{N\ge 1} \left| e_N^{\mathrm{i}0(-\Delta)^\alpha} (g-f) \right| \right\|_{L^1(\mathbb{B}^n; \mathrm{d}\mu(x))} \\
\lesssim& \lambda^{-1} \sqrt{C_\kappa(\mu)} \|f-g\|_{H^s(\mathbb{R}^n)} \\
\lesssim& \lambda^{-1} \sqrt{C_\kappa(\mu)} \epsilon.
\end{aligned} \tag{2.8}$$

Upon first letting $\epsilon \to 0$, and then letting $\lambda \to \infty$, we have

$$\mu \left\{ x \in \mathbb{B}^n : \lim_{t\to 0} e^{\mathrm{i}t(-\Delta)^\alpha} f(x) \ne f(x) \right\} = 0$$

whenever $\mu \in M^\kappa(\mathbb{B}^n)$ with $\kappa > \kappa_0$.

If $\mathbb{H}^\kappa$ denotes the κ-dimensional Hausdorff measure which is of translation invariance and countable additivity, then Frostman's lemma is used to derive that

$$\mathbb{H}^\kappa \left\{x \in \mathbb{B}^n : \lim_{t\to 0} e^{\mathrm{i}t(-\Delta)^\alpha} f(x) \neq f(x)\right\} = 0, \quad \kappa > \kappa_0.$$

Hence,

$$\mathrm{d}(s,n,\alpha) = \sup_{f\in H^s(\mathbb{R}^n)} \dim_H \left\{x \in \mathbb{R}^n : \lim_{t\to 0} e^{\mathrm{i}t(-\Delta)^\alpha} f(x) \neq f(x)\right\} \leq \kappa_0.$$

□

2.2 Proof of Theorem 1.1

We begin with a statement of the following key result, whose proof will be presented in Section 3, due to its nontriviality:

Theorem 2.2 If

$$\begin{cases} n \geq 2; \\ 0 < \kappa \leq n; \\ \mu \in M^\kappa(\mathbb{B}^n); \\ R \geq 1; \\ \mathrm{d}\mu_R(x) = R^\kappa \mathrm{d}\mu\left(\dfrac{x}{R}\right); \\ f \in H^s(\mathbb{R}^n); \\ \mathrm{supp}\hat{f} \subset A(1) = \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}, \end{cases}$$

then

$$\left\|\sup_{0<t<R} \left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} \lessapprox R^{\frac{\kappa}{2(n+1)}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.9}$$

Consequently, we have the following assertion:

Corollary 2.3 If

$$\begin{cases} n \geq 2; \\ 0 < \kappa \leq n; \\ s > \dfrac{\kappa}{2(n+1)} + \dfrac{n-\kappa}{2}; \\ \mu \in M^\kappa(\mathbb{B}^n); \\ f \in H^s(\mathbb{R}^n), \end{cases}$$

then

$$\left\|\sup_{0<t<1} \left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(x))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}. \tag{2.10}$$

Proof Employing Theorem 2.2 and its notations, as well as [1] (see [10, 11, 24, 25]), we get that

$$\left\|\sup_{0<t<R^{2\alpha}} \left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} \lessapprox R^{\frac{\kappa}{2(n+1)}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

Next, we use parabolic rescaling. More precisely, if

$$\begin{cases}\xi = R^{-1}\eta;\\ x = RX;\\ t = R^{2\alpha}T;\\ f_R(x) = f(Rx);\\ \operatorname{supp}\widehat{f_R} \subset A(R) = \{\xi \in \mathbb{R}^n : |\xi| \sim R\},\end{cases}$$

then

$$\begin{aligned}e^{\mathrm{i}t(-\Delta)^\alpha} f(x) &= \int_{\mathbb{R}^n} e^{\mathrm{i}(x\cdot\xi + t|\xi|^{2\alpha})}\hat{f}(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n} e^{\mathrm{i}(R^{-1}x\cdot\eta + tR^{-2\alpha}|\eta|^{2\alpha})}\widehat{f(R\cdot)}(\eta)\mathrm{d}\eta\\ &= \int_{\mathbb{R}^n} e^{\mathrm{i}(X\cdot\eta + T|\eta|^{2\alpha})}\widehat{f_R}(\eta)\mathrm{d}\eta = e^{\mathrm{i}T(-\Delta)^\alpha} f_R(X),\end{aligned}$$

and hence

$$\begin{cases}\left\|\sup\limits_{0<t<R^{2\alpha}}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} = R^{\frac{\kappa}{2}}\left\|\sup\limits_{0<T<1}\left|e^{\mathrm{i}T(-\Delta)^\alpha} f_R\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(X))};\\ \|f_R\|_{L^2(\mathbb{R}^n)} = \left(\displaystyle\int_{\mathbb{R}^n}|f_R(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} = R^{-\frac{n}{2}}\|f\|_{L^2(\mathbb{R}^n)};\\ R^{\frac{\kappa}{2}}\left\|\sup\limits_{0<T<1}\left|e^{\mathrm{i}T(-\Delta)^\alpha} f_R\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(X))} \lessapprox R^{\frac{\kappa}{2(n+1)}} R^{\frac{n}{2}}\|f_R\|_{L^2(\mathbb{R}^n)}.\end{cases}$$

Consequently, if $T = t$ and $X = x$, then

$$\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f_R\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(x))} \lessapprox R^{\frac{\kappa}{2(n+1)}+\frac{n-\kappa}{2}}\|f_R\|_{L^2(\mathbb{R}^n)}, \tag{2.12}$$

and hence Littlewood-Paley's decomposition yields that

$$\begin{cases} f = f_0 + \sum\limits_{k\geq 1} f_k;\\ \operatorname{supp}\widehat{f_0} \subset A(1);\\ \operatorname{supp}\widehat{f_k} \subset A(2^k) = \{\xi \in \mathbb{R}^n : |\xi| \sim 2^k\}.\end{cases}$$

Finally, by Minkowski's inequality and (2.12), as well as

$$s > \frac{\kappa}{2(n+1)} + \frac{n-\kappa}{2},$$

we arrive at

$$\begin{aligned}&\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(x))}\\ \leq &\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f_0\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(x))} + \sum_{k\geq 1}\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^\alpha} f_k\right|\right\|_{L^2(\mathbb{B}^n;\mathrm{d}\mu(x))}\\ \lesssim &\|f_0\|_{L^2(\mathbb{R}^n)} + \sum_{k\geq 1} 2^{k\left(\frac{\kappa}{2(n+1)}+\frac{n-\kappa}{2}\right)}\|f_k\|_{L^2(\mathbb{R}^n)}\\ \lesssim &\|f\|_{H^s(\mathbb{R}^n)} + \sum_{k\geq 1} 2^{k\left(\frac{\kappa}{2(n+1)}+\frac{n-\kappa}{2}-s\right)}\|f\|_{H^s(\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

$\square$

Next we use Corollary 2.3 to prove Theorem 1.1.

Proof An application of the Hölder inequality and (2.2) in Corollary 2.3 derives that

$$\left\|\sup_{0<t<1}\left|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\right|\right\|_{L^1(\mathbb{B}^n;\mathrm{d}\mu(x))}\lesssim\sqrt{C_\kappa(\mu)}\|f\|_{H^s(\mathbb{R}^n)},$$

whence (2.2) follows. Thus, Proposition 2.1 yields that

$$\mathrm{d}(s,n,\alpha)\leq\kappa_0\in[n-2s,\kappa).$$

Also, since

$$s>\frac{\kappa}{2(n+1)}+\frac{n-\kappa}{2},$$

we have that

$$n\geq\kappa>n+1-\frac{2(n+1)s}{n}.$$

Choosing

$$\kappa_0=n+1-\frac{2(n+1)s}{n}$$

implies that

$$\mathrm{d}(s,n,\alpha)\leq n+1-\frac{2(n+1)s}{n}.$$

Next, we make the following two-fold analysis:

On the one hand, we ask for

$$n+1-\frac{2(n+1)s}{n}\geq n-2s\Leftrightarrow s\leq\frac{n}{2}.$$

On the other hand, it is natural to request that

$$n+1-\frac{2(n+1)s}{n}<n\Leftrightarrow s>\frac{n}{2(n+1)}.$$

Accordingly,

$$\frac{n}{2(n+1)}<s\leq\frac{n}{2}$$

is required in the hypothesis of Theorem 1.1. □

3 Theorem 3.1 ⇒ Theorem 2.2

3.1 Theorem 3.1 ⇒ Corollary 3.2

We say that a collection of quantities are dyadically constant if all the quantities are in the same interval of the form $(2^j,2^{j+1}]$, where j is an integer. The key ingredient of the proof of Theorem 2.2 is the following, which will be proved in Section 4:

Theorem 3.1 Let

$$\begin{cases}(n,R)\in\mathbb{N}\times[1,\infty);\\ \mathrm{supp}\hat{f}\subset\mathbb{B}^n;\\ p=\dfrac{2(n+1)}{n-1}.\end{cases}$$

Then, for any $0<\epsilon<\frac{1}{100}$, there exist constants

$$C_\epsilon>0\text{ and }0<\delta=\delta(\epsilon)\ll\epsilon$$

such that if

(i) $Y=\bigcup_{k=1}^{M}B_k$ is a union of lattice K^2-cubes in $B^{n+1}(0,R)$ and each lattice $R^{\frac{1}{2}}$-cube intersecting Y contains $\sim\lambda$ many K^2-cubes in Y, where $K=R^{\delta}$;

(ii) $\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\|_{L^p(B_k)}$ is dyadically a constant in $k=1,2,\dots,M$;

(iii) $1\le\kappa\le n+1$ and γ is given by

$$\gamma=\max_{\substack{B^{n+1}(x',r)\subset B^{n+1}(0,R)\\ x'\in\mathbb{R}^{n+1},r\ge K^2}}\frac{\#\{B_k:B_k\subset B^{n+1}(x',r)\}}{r^{\kappa}},\tag{3.1}$$

then

$$\left\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\right\|_{L^p(Y)}\le C_{\epsilon}M^{-\frac{1}{n+1}}\gamma^{\frac{2}{(n+1)(n+2)}}\lambda^{\frac{n}{(n+1)(n+2)}}R^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}.\tag{3.2}$$

From Theorem 3.1, we can get the following L^2-restriction estimate:

Corollary 3.2 Let

$$(n,R)\in\mathbb{N}\times[1,\infty)\ \text{ and }\ \mathrm{supp}\hat{f}\subset\mathbb{B}^n.$$

Then, for any $\epsilon>0$, there exists a constant $C_{\epsilon}>0$ such that if

(i) $X=\bigcup_k B_k$ is a union of lattice unit cubes in $B^{n+1}(0,R)$;

(ii) $1\le\kappa\le n+1$ and γ is given by

$$\gamma=\max_{\substack{B^{n+1}(x',r)\subset B^{n+1}(0,R)\\ x'\in\mathbb{R}^{n+1},r\ge 1}}\frac{\#\{B_k:B_k\subset B^{n+1}(x',r)\}}{r^{\kappa}},\tag{3.3}$$

then

$$\left\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\right\|_{L^2(X)}\le C_{\epsilon}\gamma^{\frac{1}{n+1}}R^{\frac{\kappa}{2(n+1)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}.\tag{3.4}$$

Proof For any $1\le\lambda\le R^{O(1)}$, we introduce the notation

$$\mathscr{Z}_{\lambda}=\big\{B_k:B_k\subset X \text{ such that any } R^{\frac{1}{2}}\text{-cube contains} \sim\lambda \text{ unit cubes } B_k \text{ in it}\big\}.$$

By pigeonholing, we fix λ such that

$$\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\|_{L^2(X)}\lessapprox\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\|_{L^2(\bigcup_{B_k\in\mathscr{Z}_{\lambda}}B_k)}.$$

It is easy to see that

$$\lambda\le\gamma R^{\frac{\kappa}{2}}$$

by taking $r=R^{\frac{1}{2}}$ in (3.3).

Next, we assume that the following inequality holds (we will prove this inequality later):

$$\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\|_{L^2(\bigcup_{B_k\in\mathscr{Z}_{\lambda}}B_k)}\lessapprox\gamma^{\frac{2}{(n+1)(n+2)}}\lambda^{\frac{n}{(n+1)(n+2)}}R^{\frac{\kappa}{(n+1)(n+2)}}\|f\|_{L^2(\mathbb{R}^n)}.\tag{3.5}$$

We thereby reach

$$\left\|e^{\mathrm{i}t(-\Delta)^{\alpha}}f\right\|_{L^2(X)}\le C_{\epsilon}\gamma^{\frac{1}{n+1}}R^{\frac{\kappa}{2(n+1)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}.$$

Hence, it remains to prove (3.5).

First, denote $Z = \bigcup_{B_k \in \mathcal{Z}_\lambda} B_k$. We can sort things into at most $O(\log R)$ many subsets of Z according to the value of $\|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^p(B_k)}$. In each subset, the value of $\|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^p(B_k)}$ is dyadically a constant. Among the subsets we can find a set $Z' \subset Z$ such that

$$\{\|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^p(B_k)} : B_k \subset Z'\} \text{ are dyadically constants}$$

and

$$\|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^2(Z)} \lessapprox \|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^2(Z')}.$$

Upon writing

$$M = \#\{B : B \text{ is unit cube and } B \subset Z'\}$$

and using Hölder's inequality, we have that

$$\begin{aligned}\left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^2(Z)} &\lessapprox \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^2(Z')} \le \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Z')} |Z'|^{\frac{1}{2}-\frac{1}{p}} \\ &\le M^{\frac{1}{n+1}} \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Z')}.\end{aligned}$$

Thus, in order to prove (3.5), it suffices to prove

$$\left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Z')} \lessapprox M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\kappa}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.6}$$

In order to use the result of Theorem 3.1, we need to extend the size of the unit cube to the K^2-cube according to the following two steps:

Step 1 Let β be a dyadic number, let $\mathcal{B}_\beta := \{B : B \subset Z', \text{ and for any lattice } K^2-\text{cube } \tilde{B} \supset B \text{ such that } \|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^p(\tilde{B})} \sim \beta\}$, we set

$$\tilde{\mathcal{B}}_\beta = \{\tilde{B} : \text{the relevant } K^2\text{-cubes}\}.$$

Step 2 Next, fixing β, letting λ' be a dyadic number, and denoting

$$\begin{cases} \mathcal{B}_{\beta,\lambda'} = \{B \in \mathcal{B}_\beta : R^{\frac{1}{2}}\text{-cube } Q \text{ contains } \lambda' \text{ many } K^2\text{-cubes from } \tilde{\mathcal{B}}_\beta\}; \\ \tilde{\mathcal{B}}_{\beta,\lambda'} = \{\tilde{B} : \text{the relevant } K^2\text{-cubes}\}, \end{cases}$$

we find that the pair $\{\beta, \lambda'\}$ satisfies

$$M' = \#\tilde{\mathcal{B}}_{\beta,\lambda'} \gtrapprox M.$$

From the definitions of λ and γ, we have

$$\begin{cases} \lambda' \le \lambda; \\ \gamma' = \max\limits_{\substack{B^{n+1}(x',r) \subset B^{n+1}(0,R) \\ x' \in \mathbb{R}^{n+1}, r \ge K^2}} \dfrac{\#\{\tilde{B} : \tilde{B} \in \tilde{\mathcal{B}}_{\beta,\lambda'}, \tilde{B} \subset B^{n+1}(x', r)\}}{r^\kappa} \le \gamma. \end{cases}$$

If $Y = \bigcup_{\tilde{B} \in \tilde{\mathcal{B}}_{\beta,\lambda'}}$, then Theorem 3.1 yields

$$\begin{aligned}\left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Z')} &\lessapprox \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Y)} \\ &\lessapprox M'^{-\frac{1}{n+1}} \gamma'^{\frac{2}{(n+1)(n+2)}} \lambda'^{\frac{n}{(n+1)(n+2)}} R^{\frac{\kappa}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)} \\ &\lessapprox M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\kappa}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

which is the desired (3.6). □

3.2 Proof of Theorem 2.2

In this section, we use Corollary 3.2 to prove Theorem 2.2.

We have

$$\mathrm{supp}\hat{f} \subset \mathbb{B}^n \Rightarrow \mathrm{supp}\ (e^{\mathrm{i}t(-\Delta)^\alpha} f)^\wedge \subset \mathbb{B}^{n+1}.$$

Thus,

$$\exists\ \ \psi \in \mathcal{S}(\mathbb{R}^{n+1})\ \ \text{and}\ \ \hat{\psi} \equiv 1 \text{ on } B^{n+1}(0,2) \quad \text{such that}\ \ (e^{\mathrm{i}t(-\Delta)^\alpha} f)^2 = (e^{\mathrm{i}t(-\Delta)^\alpha} f)^2 * \psi.$$

If

$$\max_{|\tilde{y}-(x,t)|\le e^5} |\psi(\tilde{y})| = \psi_1(x,t),$$

which decays rapidly, then for any $(x,t) \in \mathbb{R}^{n+1}$,

$$\tilde{m}(x,t) = (m, m_{n+1}) = (m_1, \cdots, m_n, m_{n+1})$$

denotes the center of the unit lattice cube containing (x,t), and hence

$$\Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * |\psi|\Big)(x,t) \le \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(\tilde{m}(x,t)).$$

Accordingly,

$$\begin{aligned}
&\Big\|\sup_{0<t<R}\Big|e^{\mathrm{i}t(-\Delta)^\alpha} f\Big|\Big\|^2_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} \\
&= \int_{B^n(0,R)} \sup_{0<t<R} \Big|e^{\mathrm{i}t(-\Delta)^\alpha} f(x)\Big|^2 \mathrm{d}\mu_R(x) \\
&\le \int_{B^n(0,R)} \sup_{0<t<R} \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * |\psi|\Big)(x,t)\mathrm{d}\mu_R(x) \\
&\le \int_{B^n(0,R)} \sup_{0<t<R} \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(\tilde{m}(x,t))\mathrm{d}\mu_R(x) \\
&\le \sum_{\substack{m=(m_1,\ldots,m_n)\in\mathbb{Z}^n \\ |m_i|,|m_{n+1}|\le R}} \bigg(\int_{|x-m|\le 10} \mathrm{d}\mu_R(x)\bigg) \cdot \sup_{\substack{m\in\mathbb{Z}^n \\ 0\le m_{n+1}\le R}} \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(m, m_{n+1}).
\end{aligned} \tag{3.7}$$

For each $m \in \mathbb{Z}^n$, let $b(m)$ be an integer in $[0,R]$ such that

$$\sup_{\substack{m_{n+1}\in\mathbb{Z} \\ 0\le m_{n+1}\le R}} \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(m, m_{n+1}) = \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(m, b(m)).$$

Next, by defining

$$v_m = \int_{|x-m|\le 10} \mathrm{d}\mu_R(x) \lesssim 1,$$

and using (3.7), we have

$$\begin{aligned}
&\Big\|\sup_{0<t<R}\Big|e^{\mathrm{i}t(-\Delta)^\alpha} f\Big|\Big\|^2_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} \\
&\lesssim \sum_{\substack{v \text{ dyadic} \\ v\in[R^{-10n},1]}} \sum_{\substack{m\in\mathbb{Z}^n,|m_i|\le R \\ v_m\sim v}} v \cdot \Big(|e^{\mathrm{i}t(-\Delta)^\alpha} f|^2 * \psi_1\Big)(m, b(m)) + R^{-9n}.
\end{aligned} \tag{3.8}$$

By pigeonholing, we get that for any small $\epsilon > 0$,

$$\Big\|\sup_{0<t<R}\Big|e^{\mathrm{i}t(-\Delta)^\alpha} f\Big|\Big\|^2_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))}$$

$$
\begin{aligned}
&\lessapprox \sum_{\substack{m\in\mathbb{Z}^n,|m_i|\le R\\ v_m\sim v}} v\cdot\left(|e^{\mathrm{i}t(-\Delta)^\alpha}f|^2*\psi_1\right)(m,b(m))+R^{-8n}\\
&\lessapprox \sum_{\substack{m\in\mathbb{Z}^n,|m_i|\le R\\ v_m\sim v}} v\cdot\left(\int_{B^{n+1}((m,b(m)),R^\epsilon)}|e^{\mathrm{i}t(-\Delta)^\alpha}f|^2\right)+R^{-8n}\\
&\lessapprox v\cdot\int_{\bigcup\limits_{m\in A_v}B^{n+1}((m,b(m)),R^\epsilon)}|e^{\mathrm{i}t(-\Delta)^\alpha}f|^2+R^{-8n}. \qquad (3.9)
\end{aligned}
$$

Note that

$$
X_v=\bigcup_{m\in\mathbb{Z}^n:|m_i|\le R \text{ and } v_m\sim v} B^{n+1}((m,b(m)),R^\epsilon)
$$

is not only a union of some distinct R^ϵ-balls, but also a union of some unit balls. Thus, the projections of these balls onto the $(x_1,\dots,x_n)$-plane are essentially disjoint (a point can be covered $\lesssim R^\epsilon$ times). For every $r>R^{2\epsilon}$, the definition of $\{m\in\mathbb{Z}^n:|m_i|\le R \text{ and } v_m\sim v\}$ ensures that the intersection of X_v and any r-ball can be contained in $\lesssim R^{10n\epsilon}v^{-1}r^\kappa$ disjoint R^ϵ- balls. Hence we can apply Corollary 3.2 to X_v with

$$
\gamma\lesssim R^{100n\epsilon}v^{-1} \quad\text{and}\quad 1\le\kappa\le n+1.
$$

By (3.9), we reach (2.9) via

$$
\begin{aligned}
\left\|\sup_{0<t<R}\left|e^{\mathrm{i}t(-\Delta)^\alpha}f\right|\right\|^2_{L^2(B^n(0,R);\mathrm{d}\mu_R(x))} &\lesssim v\left(\gamma^{\frac{1}{n+1}}R^{\frac{\kappa}{2(n+1)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}\right)^2\\
&\lessapprox v^{\frac{n-1}{n+1}}R^{\frac{\kappa}{n+1}}\|f\|^2_{L^2(\mathbb{R}^n)}\\
&\lessapprox R^{\frac{\kappa}{n+1}}\|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

□

4 Conclusion

4.1 Proof of Theorem 3.1 - $R\lesssim 1$

In what follows, we always assume that

$$
\begin{cases}
p=\dfrac{2(n+1)}{n-1};\\
q=\dfrac{2(n+2)}{n};\\
\mathrm{supp}\hat{f}\subset\mathbb{B}^n.
\end{cases}
$$

Nevertheless, estimate (3.2) under $R\lesssim 1$ is trivial. In fact, from the assumptions of Theorem 3.1, we see that

$$
M\sim\lambda\sim\gamma\sim R\sim 1.
$$

Furthermore, by the short-time Strichartz estimate (see [26, 27]), we get that

$$
\left\|e^{\mathrm{i}t(-\Delta)^\alpha}f\right\|_{L^p(Y)}\le\left\|e^{\mathrm{i}t(-\Delta)^\alpha}f\right\|_{L^p([0,1]\times\mathbb{R}^n)}\lesssim\|f\|_{L^2(\mathbb{R}^n)}, \qquad (4.1)
$$

thereby verifying Theorem 3.1 for $R\lesssim 1$.

4.2 Proof of Theorem 3.1 - $R \gg 1$

First, we decompose the unit ball in the frequency space into disjoint K^{-1}-cubes τ. Write

$$\begin{cases} \mathcal{S} = \{\tau : K^{-1} - \text{cubes } \tau \subset \mathbb{B}^n\}; \\ f = \sum_\tau f_\tau; \\ \widehat{f_\tau} = \hat{f}\chi_\tau; \\ \mathcal{S}(B) = \left\{\tau \in \mathcal{S} : \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f_\tau\right\|_{L^p(B)} \ge \dfrac{1}{100(\#\mathcal{S})}\left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(B)}\right\} \quad \text{for a } K^2 - \text{cube } B. \end{cases}$$

Then,

$$\left\|\sum_{\tau\in\mathcal{S}(B)} e^{\mathrm{i}t(-\Delta)^\alpha} f_\tau\right\|_{L^p(B)} \sim \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(B)}.$$

Second, we recall the definitions of narrow cubes and broad cubes.

We say that a K^2-cube B is narrow if there is an n-dimensional subspace V such that for all $\tau \in \mathcal{S}(B)$,

$$\angle(G(\tau), V) \le \frac{1}{100nK},$$

where $G(\tau) \subset \mathbb{S}^n$ is a spherical cap of radius $\sim K^{-1}$ given by

$$G(\tau) = \left\{\frac{(-2\xi, 1)}{|(-2\xi, 1)|} \in \mathbb{S}^n : \xi \in \tau\right\},$$

and $\angle(G(\tau), V)$ denotes the smallest angle between any non-zero vector $v \in V$ and $v' \in G(\tau)$.

Otherwise, we say that the K^2-cube B is broad. In other words, a cube being broad means that the tiles $\tau \in \mathcal{S}(B)$ are so separated that the norm vectors of the corresponding spherical caps cannot be in an n-dimensional subspace; more precisely, for any broad B,

$$\exists\ \tau_1, \dots, \tau_{n+1} \in \mathcal{S}(B) \ \text{ such that } \ |v_1 \wedge v_2 \wedge \dots \wedge v_{n+1}| \gtrsim K^{-n} \ \ \forall \ \ v_j \in G(\tau_j). \tag{4.2}$$

Third, with the setting

$$\begin{cases} Y_{\text{broad}} = \bigcup_{B_k \text{ is broad}} B_k; \\ Y_{\text{narrow}} = \bigcup_{B_k \text{ is narrow}} B_k, \end{cases}$$

we will handle Y according to the sizes of Y_{broad} and Y_{narrow}. Thus,

(1) We call the case broad if Y_{broad} contains $\ge \frac{M}{2}$ many K^2-cubes, and we will deal with the broad case using the multilinear refined Strichartz estimates.

(2) We call the case narrow if Y_{narrow} contains $\ge \frac{M}{2}$ many K^2-cubes, and we will handle the narrow case by l^2-decoupling, parabolic rescaling and induction on scales.

4.2.1 The broad case

In this case, we consider the same generalized Schrödinger operators as Cho-Ko [1]. The idea here is to take the case as a close perturbation of the typical curve $|\xi|^2$ in a very small scale and to keep this perturbation under parabolic scaling. This cannot be true for $|\xi|^{2\alpha}$ with $\alpha > \frac{1}{2}$, but it is true for its quadratic term. This is the reason for introducing the set $\mathcal{NPF}(L, c_0)$ and for applying induction in this set. Let us recall the two definitions in [1].

Let $\Phi(D)$ be a multiplier operator defined on $\mathbb{R}^n$ which satisfies

$$\begin{cases}\Phi(\xi) \text{ is smooth at } \xi \neq 0;\\ |D^\beta \Phi(\xi)| \lesssim |\xi|^{2\alpha-|\beta|} \ \& \ |\nabla\Phi(\xi)| \gtrsim |\xi|^{2\alpha-1} \ \forall \text{ multi-index } \beta;\\ \text{The Hessian matrix of } \Phi \text{ is positive definite.}\end{cases} \tag{4.3}$$

Let $0 < c_0 \ll 1$ and $L \in \mathbb{N}$ be sufficiently large. We consider a collection of the normalized phase functions as follows:

$$\mathcal{NPF}(L, c_0) = \left\{ \Phi \in C_0^\infty(B^n(0,2)) : \left\| \Phi(\xi) - \frac{|\xi|^2}{2} \right\|_{C^L(\mathbb{B}^n)} \leq c_0 \right\}.$$

Theorem 4.1 (Linear refined Strichartz estimate in dimension $n+1$) Suppose that

(i) Φ is in $\mathcal{NPF}(L, c_0)$ for sufficiently small $c_0 > 0$;

(ii) $\{Q_j\}$ is a sequence of the lattice $R^{\frac{1}{2}}$-cubes in $B^{n+1}(0,R)$ with $\|e^{\mathrm{i}t\Phi}f\|_{L^q(Q_j)}$ being essentially constant in j;

(iii) $\{Q_j\}$ is arranged in horizontal slabs of the form $\mathbb{R}\times\ldots\times\mathbb{R}\times\{t_0, t_0+R^{\frac{1}{2}}\}$, which contains $\sim \sigma$ cubes Q_j.

Then

$$\|e^{\mathrm{i}t\Phi}f\|_{L^q(\bigcup_j Q_j)} \leq C_\epsilon R^\epsilon \sigma^{-\frac{1}{n+2}} \|f\|_{L^2(\mathbb{R}^n)} \quad \forall \ \ \epsilon > 0. \tag{4.4}$$

Remark 4.2 On the one hand, by taking $\Phi(\xi) = |\xi|^2$, we can rediscover the results for the Schrödinger operator by Du-Guth-Li [19] in $\mathbb{R}^{2+1}$, and in higher dimensional cases [5]. Similar results can also be found in [1], with an extra restriction condition on the support of f.

On the other hand, for $\Phi(\xi) = |\xi|^{2\alpha}$ with $\alpha > \frac{1}{2}$, we can reduce Φ satisfying (4.3) to a function in $\mathcal{NPF}(L, c_0)$. Denote by $H\Phi(\xi_0)$ the Hessian matrix of $\Phi(\xi)$ at point ξ_0. Since the Hessian matrix of Φ is positive definite, we can write it as $H\Phi(\xi_0) = P^{-1}DP$, with P a symmetric matrix $D = (\lambda_1\mathbf{e}_1, \ldots, \lambda_n\mathbf{e}_n)$ and $\lambda_1 > 0, \ldots, \lambda_n > 0$. We introduce a new function around point ξ_0:

$$\Phi_{\rho,\xi_0}(\xi) = \rho^{-2}\left(\Phi(\rho H^{-1}\xi + \xi_0) - \Phi(\xi_0) - \rho\nabla\Phi(\xi_0)\cdot H^{-1}\xi\right). \tag{4.5}$$

From Cho-Ko [1], we have $\Phi_{\rho,\xi_0} \in \mathcal{NPF}(L, c_0)$ for a sufficiently small $\rho = \rho(\Phi, L, c_0) > 0$. Moreover,

$$\begin{aligned}\left|e^{\mathrm{i}t\Phi}f(x)\right| &= \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{\mathrm{i}(x,t)\cdot(\xi,\Phi(\xi))}\hat{f}(\xi)\mathrm{d}\xi\right| \\ &= \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{\mathrm{i}(x,t)\cdot(\rho H^{-1}\eta+\xi_0,\Phi(\rho H^{-1}\eta+\xi_0))}\hat{f}(\rho H^{-1}\eta+\xi_0)\rho^n|H|^{-1}\mathrm{d}\eta\right| \\ &= \rho^n|H|^{-1}(2\pi)^{-n}\left|\int_{\mathbb{R}^n} e^{\mathrm{i}(\rho H^{-t}x+\rho t H^{-t}\nabla\Phi(\xi_0),\rho^2 t)\cdot(\eta,\Phi_{\rho,\xi_0}(\eta))}\hat{f}(\rho H^{-1}\eta+\xi_0)\mathrm{d}\eta\right|.\end{aligned}$$

Next, we use

$$\begin{cases} x' = \rho H^{-t}(x + t\nabla\Phi(\xi_0));\\ t' = \rho^2 t;\\ \widehat{f}_{\rho,\xi_0}(\eta) = \rho^{\frac{n}{2}}|H|^{-\frac{1}{2}}\hat{f}(\rho H^{-1}\eta+\xi_0);\\ \|f\|_{L^2(\mathbb{R}^n)} = \|f_{\rho,\xi_0}\|_{L^2(\mathbb{R}^n)}\end{cases}$$

to get that

$$\begin{aligned}
&\left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(S)}^q=\int_S|e^{\mathrm{i}t\Phi}f(x)|^q\mathrm{d}x\mathrm{d}t\\
&=\int_S\left|\rho^n|H|^{-1}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}(\rho H^{-t}x+\rho tH^{-t}\nabla\Phi(\xi_0),\rho^2t)\cdot(\eta,\Phi_{\rho,\xi_0}(\eta))}\hat{f}(\rho H^{-1}\eta+\xi_0)\mathrm{d}\eta\right|^q\mathrm{d}x\mathrm{d}t\\
&=\rho^{nq}|H|^{-q}\int_{S'}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}(x',t')\cdot(\eta,\Phi_{\rho,\xi_0}(\eta))}\hat{f}(\rho H^{-1}\eta+\xi_0)\mathrm{d}\eta\right|^q\rho^{-n}|H|\mathrm{d}x'\rho^{-2}\mathrm{d}t'\\
&=\rho^{nq-n-2-\frac{nq}{2}}|H|^{-q+1+\frac{q}{2}}\int_{S'}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}(x',t')\cdot(\eta,\Phi_{\rho,\xi_0}(\eta))}\rho^{\frac{n}{2}}|H|^{-\frac12}\right.\\
&\quad\left.\times\hat{f}(\rho H^{-1}\eta+\xi_0)\mathrm{d}\eta\right|^q\mathrm{d}x'\mathrm{d}t'\\
&=\rho^{\frac{nq}{2}-n-2}|H|^{-\frac{q}{2}+1}\int_{S'}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}(x',t')\cdot(\eta,\Phi_{\rho,\xi_0}(\eta))}\hat{f}_{\rho,\xi_0}(\eta)\mathrm{d}\eta\right|^q\mathrm{d}x'\mathrm{d}t'\\
&=\rho^{\frac{nq}{2}-n-2}|H|^{-\frac{q}{2}+1}\left\|e^{\mathrm{i}t'\Phi_{\rho,\xi_0}}f_{\rho,\xi_0}\right\|_{L^q(S')}^q.
\end{aligned}$$

In short, we have

$$\left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(S)}=\rho^{\frac{n}{2}-\frac{n+2}{q}}|H|^{\frac1q-\frac12}\left\|e^{\mathrm{i}t'\Phi_{\rho,\xi_0}}f_{\rho,\xi_0}\right\|_{L^q(S')}. \tag{4.6}$$

Note that

$$\frac{n}{2}-\frac{n+2}{q}=0\ \text{ and }\ |H|\sim1\ (\text{since }\operatorname{supp}\hat{f}\subset\{\xi:|\xi|\sim1\}),$$

and the change of variables does not change the value of σ. Thus, (4.4) is also true for the generalized phase functions Φ satisfying (4.3), which contains $\Phi(\xi)=|\xi|^{2\alpha}$ with $\alpha>\frac12$.

Lemma 4.3 (Bourgain-Demeter's l^2-decoupling inequality [28]) Suppose that $\hat{g}$ is supported in a σ-neighborhood of an elliptic surface S in $\mathbb{R}^n$. If τ is a rectangle of size $\sigma^{\frac12}\times\dots\times\sigma^{\frac12}\times\sigma$ inside the σ-neighborhood of S, $\widehat{g_\tau}=\hat{g}\chi_\tau$ and $\epsilon>0$, then

$$\|g\|_{L^p(\mathbb{R}^n)}\le C_\epsilon\sigma^{-\epsilon}\left(\sum_\tau\|g_\tau\|_{L^p(\mathbb{R}^n)}^2\right)^{\frac12}.$$

Next we begin the proof of Theorem 4.1.

Proof We prove a linear refined Strichartz estimate in dimension $n+1$ by four steps.

First, we consider the wave packet decomposition of f. For any smooth function $f:\mathbb{B}^n\to\mathbb{R}$, we decompose it into wave packets, and each wave packet is supported in a ball θ of radius $R^{-\frac14}$. Then we divide the physical space $B^n(0,R)$ into balls D of radius $R^{\frac34}$. From [29], we have that

$$f=\sum_{\theta,D}f_{T_{\theta,D}}\ \text{ and }\ f_{T_{\theta,D}}=(\hat{f}\chi_\theta)^\vee\chi_D,$$

and we have that the functions $f_{T_{\theta,D}}$ are approximately orthogonal, thereby giving us

$$\|f\|_{L^2(\mathbb{R}^n)}^2\sim\sum_{\theta,D}\|f_{T_{\theta,D}}\|_{L^2(\mathbb{R}^n)}^2.$$

By computation, we have that the restriction of $e^{\mathrm{i}t\Phi}f_{T_{\theta,D}}(x)$ to $B^{n+1}(0,R)$ is essentially supported on a tube $T_{\theta,D}$, which is defined as follows:

$$T_{\theta,D}=\Big\{(x,t):(x,t)\in B^{n+1}(0,R)\ \text{ and }\ |x-c(D)-t\nabla\Phi(c(\theta))|\le R^{\frac34+\delta}\ \ \&\ \ 0<t<R\Big\}.$$

Here $c(\theta)$ & $c(D)$ denote the centers of θ and D, respectively. Therefore, by a decoupling theorem, we have that

$$\left\|e^{\mathrm{it}\Phi}f\right\|_{L^q(Q)} \lessapprox \left(\sum_T \left\|e^{\mathrm{it}\Phi}f_T\right\|^2_{L^q(Q)}\right)^{\frac{1}{2}},$$

where $T_{\theta,D}=T$. In fact, we take $\eta_Q\in\mathcal{S}(\mathbb{R}^{n+1})$ such that supp $\widehat{\eta_Q}\subset Q^*$ and Q^* is a $R^{-\frac{1}{2}}$-cube. We have $|\eta_Q|\sim 1$ on Q. By Lemma 4.3, we obtain

$$\left\|e^{\mathrm{it}\Phi}f\right\|_{L^q(Q)} \lesssim \left\|e^{\mathrm{it}\Phi}f\eta_Q\right\|_{L^q(\mathbb{R}^{n+1})} \lesssim \left(\sum_T \left\|e^{\mathrm{it}\Phi}f_T\eta_Q\right\|^2_{L^q(\mathbb{R}^{n+1})}\right)^{\frac{1}{2}} \lessapprox \left(\sum_T \left\|e^{\mathrm{it}\Phi}f_T\right\|^2_{L^q(Q)}\right)^{\frac{1}{2}}.$$

Second, we use parabolic rescaling and an induction on radius $R^{\frac{1}{2}}$. Suppose that

$\{S_j\}_j$ are $R^{\frac{1}{2}}\times\ldots\times R^{\frac{1}{2}}\times R^{\frac{3}{4}}$-tubes in T, which are parallel to the long axes of T;

$\|e^{\mathrm{it}\Phi}f_T\|_{L^q(S_j)}$ is essentially dyadically constant in j;

these tubes are arranged into $R^{\frac{3}{4}}$-slabs running parallel to the short axes of T, which contains $\sim\sigma_T$ tubes S_j;

$Y_T=\bigcup_j S_j$.

Then,

$$\left\|e^{\mathrm{it}\Phi}f_T\right\|_{L^q(Y_T)} \le C_\epsilon R^{\frac{\epsilon}{2}}\sigma_T^{-\frac{1}{n+2}}\|f_T\|_{L^2(\mathbb{R}^n)}. \tag{4.7}$$

In fact, as in Remark 4.2, we get that

$$\begin{cases}\left\|e^{\mathrm{it}\Phi}f\right\|_{L^q(S)} = \rho^{\frac{n}{2}-\frac{n+2}{q}}|H|^{\frac{1}{q}-\frac{1}{2}}\left\|e^{\mathrm{it}'\Phi_{\rho,\xi_0}}f_{\rho,\xi_0}\right\|_{L^q(S')};\\ \widehat{f}_{\rho,\xi_0}(\eta)=\rho^{\frac{n}{2}}|H|^{-\frac{1}{2}}\hat{f}(\rho H^{-1}\eta+\xi_0);\\ \|f\|_{L^2(\mathbb{R}^n)}=\|f_{\rho,\xi_0}\|_{L^2(\mathbb{R}^n)}.\end{cases} \tag{4.8}$$

If

$$\rho=R^{-\frac{1}{4}} \;\;\&\;\; \xi_0=c(D) \;\;\&\;\; S=Y_T \;\;\&\;\; S'=\widetilde{Y},$$

then $\widetilde{Y}$, as the image of Y_T under the new coordinate, is a union of $R^{\frac{1}{4}}$-cubes inside an $R^{\frac{1}{2}}$-cube. These $R^{\frac{1}{4}}$-cubes are arranged in $R^{\frac{1}{4}}$-horizontal slabs, and

$$\#\{R^{\frac{1}{4}}-\text{cubes}: R^{\frac{1}{4}}-\text{cubes are arranged in } R^{\frac{1}{4}}-\text{horizontal slabs}\}\sim\sigma_T,$$

hence,

$$\left\|e^{\mathrm{it}\Phi}f\right\|_{L^q(Y_T)} = |H|^{-\frac{1}{n+2}}\left\|e^{\mathrm{it}'\Phi_{\rho,\xi_0}}f_{\rho,\xi_0}\right\|_{L^q(\widetilde{Y})}.$$

By induction, we have that

$$\left\|e^{\mathrm{it}'\Phi_{\rho,\xi_0}}f_{\rho,\xi_0}\right\|_{L^q(\widetilde{Y})} \le C_\epsilon R^{\frac{\epsilon}{2}}\sigma_T^{-\frac{1}{n+2}}\|f_{\rho,\xi_0}\|_{L^2(\mathbb{R}^n)},$$

thereby giving us that, if $f=f_T$,

$$\left\|e^{\mathrm{it}\Phi}f_T\right\|_{L^q(Y_T)} \le C_\epsilon|H|^{-\frac{1}{n+2}}R^{\frac{\epsilon}{2}}\sigma_T^{-\frac{1}{n+2}}\|f_T\|_{L^2(\mathbb{R}^n)} \lesssim R^{\frac{\epsilon}{2}}\sigma_T^{-\frac{1}{n+2}}\|f_T\|_{L^2(\mathbb{R}^n)}$$

(thanks to $|H|\sim 1$) , namely that, (4.7) holds.

Third, we shall choose an appropriate Y_T. For each T, we classify tubes in T in the following ways:

For each dyadic number λ, we define $\mathbb{S}_\lambda=\Big\{S_j: S_j\subset T \;\;\&\;\; \left\|e^{\mathrm{it}\Phi}f_T\right\|_{L^q(S_j)}\sim\lambda\Big\}$.

For any dyadic number η, we define $\mathbb{S}_{\lambda,\eta} = \Big\{S_j : S_j \in \mathbb{S}_\lambda \ \ \& \ \ \#\{S_j, S_j \subset R^{\frac{3}{4}} - \text{slab}\} \sim \eta\Big\}$. We denote

$$Y_{T,\lambda,\eta} = \bigcup_{S_j \in \mathbb{S}_{\lambda,\eta}} S_j,$$

thereby getting that

$$e^{\mathrm{it}\Phi} f = \sum_{\lambda,\eta} \left(\sum_T e^{\mathrm{it}\Phi} f_T \cdot \chi_{Y_{T,\lambda,\eta}} \right).$$

For each λ, η, there are $O(\log R)$ choices. By pigeonholing, we can choose λ, η so that

$$\left\|e^{\mathrm{it}\Phi} f\right\|_{L^q(Q_j)} \lesssim (\log R)^2 \left\| \sum_T e^{\mathrm{it}\Phi} f_T \cdot \chi_{Y_{T,\lambda,\eta}} \right\|_{L^q(Q_j)}$$

holds for ≈ 1 of all cubes $Q_j \subset Y$, where $Y = \bigcup\limits_j Q_j$. In fact, we have $\#\{Q_j\}_j \lesssim R^{\frac{n+1}{2}}$ and $\#\{\lambda, \eta\} \lesssim \log R$. Since $\log R \ll R^{\frac{n+1}{2}}$, this inequality holds for ≈ 1 of all cubes $Q_j \subset Y$. Here (λ, η) is independent of Q_j.

First of all, we fix λ, η in the sequel of the proof of the refined Strichartz estimate in dimension $n+1$. Let $Y_{T,\lambda,\eta} = Y_T$ for convenience. Note that Y_T satisfies the hypotheses for our inductive estimate, where $\sigma_T = \eta$. By the definitions of Y_T and σ_T and the direction of T, we have that Y_T contains $\lesssim \sigma_T$ cubes of Q_j in any $R^{\frac{1}{2}}$-horizontal slab. Therefore,

$$|Y_T \cap Y| \lesssim \frac{\sigma_T}{\sigma} |Y|. \tag{4.9}$$

Next, we choose the tubes Y according to the dyadic size of $\|f_T\|_{L^2(\mathbb{R}^n)}$. We can restrict matters to $O(\log R)$ choices of this dyadic size, so we can choose a set of T's with $\mathbb{T}$ such that

$$\|f_T\|_{L^2(\mathbb{R}^n)} \text{ is essentially constant}$$

and

$$\left\|e^{\mathrm{it}\Phi} f\right\|_{L^q(Q_j)} \lessapprox \left\| \sum_{T\in\mathbb{T}} e^{\mathrm{it}\Phi} f_T \cdot \chi_{Y_T} \right\|_{L^q(Q_j)} \quad \text{holds for} \approx 1 \text{ of all cubes } Q_j \subset Y. \tag{4.10}$$

Lastly, we choose the cubes $Q_j \subset Y$ according to the number of Y_T that contain them. Denote that

$$Y' = \{Q_j : Q_j \subset Y, \text{ which obeys } (4.10), \text{ and each } Q_j \text{ lies in } \sim \nu \text{ of the sets } \{Y_T\}_{T\in\mathbb{T}}\}.$$

Because (4.10) holds for ≈ 1 cubes and ν are dyadic numbers, we can use (4.9) to get

$$|Y'| \approx |Y| \ \ \& \ \ |Y_T \cap Y'| \le |Y_T \cap Y| \lesssim \frac{\sigma_T}{\sigma}|Y| \approx \frac{\sigma_T}{\sigma}|Y'|,$$

thereby finding that

$$\nu \lessapprox \frac{\sigma_T}{\sigma}|\mathbb{T}|. \tag{4.11}$$

Fourth, we combine all of our ingredients and finish our proof of Theorem 4.1.

By (4.10) and the decoupling, as well as Hölder's inequality, we have that if $Q_j \subset Y'$, then

$$\left\|e^{\mathrm{it}\Phi} f\right\|_{L^q(Q_j)} \lessapprox \nu^{\frac{1}{n+2}} \left(\sum_{T\in\mathbb{T}: Q_j \subset Y_T} \left\|e^{\mathrm{it}\Phi} f_T\right\|^q_{L^q(Q_j)} \right)^{\frac{1}{q}}.$$

By making a sum over $Q_j \subset Y'$ and using our inductive hypothesis at scale $R^{\frac{1}{2}}$, we obtain that

$$\left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(Y')}^q \lessapprox \nu^{\frac{2}{n}} \sum_{T\in\mathbb{T}} \left\|e^{\mathrm{i}t\Phi}f_T\right\|_{L^q(Y_T)}^q \lessapprox \nu^{\frac{2}{n}} \sum_{T\in\mathbb{T}} \left(\sigma_T^{-\frac{1}{n+2}}\|f_T\|_{L^2(\mathbb{R}^n)}\right)^q$$
$$= \nu^{\frac{2}{n}} \sum_{T\in\mathbb{T}} \sigma_T^{-\frac{2}{n}}\|f_T\|_{L^2(\mathbb{R}^n)}^q.$$

For each $Q_j \subset Y$, since

$$\|e^{\mathrm{i}t\Phi}f\|_{L^q(Q_j)} \text{ is essentially constant in } j \text{ and } |Y'| \approx |Y|,$$

we get that

$$\|e^{\mathrm{i}t\Phi}f\|_{L^q(Y)} \approx \|e^{\mathrm{i}t\Phi}f\|_{L^q(Y')},$$

thereby utilizing (4.11) and the fact that $\|f_T\|_{L^2(\mathbb{R}^n)}$ is essentially constant among all $T \in \mathbb{T}$ to derive that

$$\left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(Y)}^q \approx \left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(Y')}^q \lessapprox \nu^{\frac{2}{n}} \sum_{T\in\mathbb{T}} \sigma_T^{-\frac{2}{n}}\|f_T\|_{L^2(\mathbb{R}^n)}^q$$
$$\lessapprox \sigma^{-\frac{2}{n}}|\mathbb{T}|^{\frac{2}{n}} \sum_{T\in\mathbb{T}} \|f_T\|_{L^2(\mathbb{R}^n)}^q \sim \sigma^{-\frac{2}{n}} \left(\sum_{T\in\mathbb{T}} \|f_T\|_{L^2(\mathbb{R}^n)}^2\right)^{\frac{n+2}{n}}$$
$$\le \sigma^{-\frac{2}{n}}\|f\|_{L^2(\mathbb{R}^n)}^q.$$

Taking the q-th root in the last estimation produces

$$\left\|e^{\mathrm{i}t\Phi}f\right\|_{L^q(Y)} \lessapprox \sigma^{-\frac{1}{n+2}}\|f\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad Y = \bigcup_j Q_j.$$

□

Moreover, Theorem 4.1 can be extended to the following form, which can be verified by [22] and Theorem 4.1:

Theorem 4.4 (Multilinear refined Strichartz estimate in dimension $n+1$.) For $2 \le k \le n+1$ and $1 \le i \le k$, let $f_i : \mathbb{R}^n \to \mathbb{C}$ have frequencies k-transversely supported in $\mathbb{B}^n$, that is,

$$1 \lesssim |\wedge_{i=1}^k G(\xi_i)| \;\;\&\;\; G(\xi_i) = \frac{(-2\xi_i, 1)}{|(-2\xi_i, 1)|} \in \mathbb{S}^n \;\;\forall\;\; \xi_i \in \mathrm{supp}\widehat{f_i}.$$

Suppose that $Q_1, Q_2, \ldots, Q_N$ are lattice $R^{\frac{1}{2}}$-cubes in $B^{n+1}(0,R)$ so that each $\|e^{\mathrm{i}t(-\Delta)^\alpha}f_i\|_{L^q(Q_j)}$ is essentially dyadically constant in j. If $Y = \bigcup\limits_{j=1}^N Q_j$ and $\epsilon > 0$, then

$$\left\|\prod_{i=1}^k \left|e^{\mathrm{i}t(-\Delta)^\alpha}f_i\right|^{\frac{1}{k}}\right\|_{L^q(Y)} \le C_\epsilon R^\epsilon N^{-\frac{k-1}{k(n+2)}} \prod_{i=1}^k \|f_i\|_{L^2(\mathbb{R}^n)}^{\frac{1}{k}}.$$

Next, we prove the broad case of Theorem 3.1.

Proof In the broad case, there are $\ge \frac{M}{2}$ many broad K^2-cubes B. Denote the collection of $(n+1)$-tuple of transverse caps by Γ as follows:

$$\Gamma = \left\{\tilde{\tau} = (\tau_1, \ldots, \tau_{n+1}) : \tau_j \in \mathcal{S} \;\;\&\;\; (4.2) \text{ holds for any } v_j \in G(\tau_j)\right\}.$$

Then, for each $B \in Y_{\rm broad}$,

$$\left\|e^{{\rm i}t(-\Delta)^\alpha} f\right\|^p_{L^p(B)} \leq K^{O(1)} \prod_{j=1}^{n+1} \left(\int_B \left|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j}\right|^p\right)^{\frac{1}{n+1}} \quad \text{for some} \quad \tilde{\tau} = (\tau_1, \dots, \tau_{n+1}) \in \Gamma.$$

In order to exploit the transversality and to make good use of the locally constant property, we break B into small balls as follows:

We cover $B = B^{n+1}(c(B), K^2)$ by cubes $B = B^{n+1}(c(B) + v, 2)$, where $v \in B^{n+1}(0, K^2) \cap \mathbb{Z}^{n+1}$. By the locally constant property, we can choose $v_j \in B^{n+1}(0, K^2) \cap \mathbb{Z}^{n+1}$ such that $\|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j}\|_{L^\infty(B)}$ is attained in $B^{n+1}(c(B) + v_j, 2)$. Writing

$$v_j = (x_j, t_j) \quad \text{and} \quad \widehat{f_{\tau_j, v_j}}(\xi) = \widehat{f_{\tau_j}}(\xi) e^{{\rm i}(x_j \cdot \xi + t_j |\xi|^{2\alpha})},$$

we deduce that

$$e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j, v_j}(x) = e^{{\rm i}(t+t_j)(-\Delta)^\alpha} f_{\tau_j}(x + x_j),$$

and $|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j, v_j}(x)|$ reaches $\|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j}\|_{L^\infty(B)}$ in $B^{n+1}(c(B), 2)$. Therefore,

$$\int_B \left|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j}\right|^p \leq K^{O(1)} \int_{B^{n+1}(c(B),2)} \left|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j, v_j}\right|^p.$$

Now, for each broad B, we find some

$$\tilde{\tau} = (\tau_1, \dots, \tau_{n+1}) \in \Gamma \ \ \& \ \ \tilde{v} = (v_1, \dots, v_{n+1})$$

such that

$$\begin{aligned} \left\|e^{{\rm i}t(-\Delta)^\alpha} f\right\|^p_{L^p(B)} &\leq K^{O(1)} \prod_{j=1}^{n+1} \left(\int_{B^{n+1}(c(B),2)} \left|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j, v_j}\right|^p\right)^{\frac{1}{n+1}} \\ &\leq K^{O(1)} \int_{B^{n+1}(c(B),2)} \prod_{j=1}^{n+1} \left|e^{{\rm i}t(-\Delta)^\alpha} f_{\tau_j, v_j}\right|^{\frac{p}{n+1}}. \end{aligned} \tag{4.12}$$

Since $\#\{\tilde{\tau}\} \lesssim K^{O(1)}$ and $\#\{\tilde{v}\} \lesssim K^{O(1)}$, we can choose some $\tilde{\tau}$ and $\tilde{v}$ such that (4.12) holds for $\geq K^{-C}M$ broad balls B. Next, we fix $\tilde{\tau}$ and $\tilde{v}$, and let $f_{\tau_j, v_j} = f_j$. Then we further sort the collection $\mathcal{B}$ of remaining broad balls as follows:

For a dyadic number A, let

$$\mathcal{B}_A = \left\{ B : B \in \mathcal{B} \text{ and for each } B \text{ we have} \left\|\prod_{j=1}^{n+1} \left|e^{{\rm i}t(-\Delta)^\alpha} f_j\right|^{\frac{1}{n+1}}\right\|_{L^\infty(B^{n+1}(c(B),2))} \sim A \right\}.$$

Fixing A, for dyadic numbers $\tilde{\lambda}_{l_1, \dots, l_{n+1}}$, let $\mathcal{B}_{A, \tilde{\lambda}_{l_1, \dots, l_{n+1}}}$ consist of all $B \in \mathcal{B}_A$ for which $R^{\frac{1}{2}}$-cube $Q \supset B$ contains $\sim \tilde{\lambda}$ cubes from $\mathcal{B}_A$ and obeys $\left\|e^{{\rm i}t(-\Delta)^\alpha} f_j\right\|_{L^q(Q)} \sim l_j$ for $j = 1, 2, \dots, n+1$.

Without loss of generality, we may assume that $\|f\|_{L^2(\mathbb{R}^n)} = 1$, and we can also assume that all of the above dyadic numbers are between R^{-C} and R^C, where C is a large constant. Therefore, there exist some dyadic numbers $A, \tilde{\lambda}_{l_1, \dots, l_{n+1}}$ such that $\#\mathcal{B}_{A, \tilde{\lambda}_{l_1, \dots, l_{n+1}}} \geq K^{-C}M$. Fix $A, \tilde{\lambda}_{l_1, \dots, l_{n+1}}$ and set $\mathcal{B}_{A, \tilde{\lambda}_{l_1, \dots, l_{n+1}}} = \mathcal{B}$. Then, by (4.12) and the definition of $\mathcal{B}_A$, we have that

$$\left\|e^{{\rm i}t(-\Delta)^\alpha} f\right\|_{L^p(Y)} \leq K^{O(1)} \left\|\prod_{j=1}^{n+1} \left|e^{{\rm i}t(-\Delta)^\alpha} f_j\right|^{\frac{1}{n+1}}\right\|_{L^p(\bigcup\limits_{B \in \mathcal{B}} B^{n+1}(c(B),2))}$$

$$
\begin{aligned}
&\le K^{O(1)} M^{\frac{1}{p}-\frac{1}{q}} \left\| \prod_{j=1}^{n+1} \left| e^{\mathrm{i}t(-\Delta)^\alpha} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\bigcup\limits_{B\in\mathcal{B}} B^{n+1}(c(B),2))} \\
&\le K^{O(1)} M^{-\frac{1}{(n+1)(n+2)}} \left\| \prod_{j=1}^{n+1} \left| e^{\mathrm{i}t(-\Delta)^\alpha} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\bigcup\limits_{Q\in\mathcal{Q}} Q)}, \qquad (4.13)
\end{aligned}
$$

where $\mathcal{Q} = \{Q:$ the relevant $R^{\frac{1}{2}}-$cubes Q defining $\mathcal{B}\}$. Note that

$$
\begin{cases}
(\#\mathcal{Q})\lambda \ge (\#\mathcal{Q})\tilde{\lambda} \sim \#\mathcal{B} \ge K^{-C}M; \\
\tilde{N} = \#\mathcal{Q} \ge \dfrac{K^{-C}M}{\lambda}.
\end{cases}
$$

Thus, by Theorem 4.4, we get

$$
\left\| \prod_{j=1}^{n+1} \left| e^{\mathrm{i}t(-\Delta)^\alpha} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\bigcup\limits_{Q\in\mathcal{Q}} Q)} \le K^{O(1)} \left(\frac{M}{\lambda}\right)^{-\frac{n}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)},
$$

thereby getting, by (4.13), that

$$
\begin{aligned}
\left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|_{L^p(Y)} &\le K^{O(1)} M^{-\frac{1}{(n+1)(n+2)}} K^{O(1)} \left(\frac{M}{\lambda}\right)^{-\frac{n}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)} \\
&\le K^{O(1)} M^{-\frac{1}{n+2}} \lambda^{\frac{n}{(n+1)(n+2)}} \|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Our goal is to prove

$$
\left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|_{L^p(Y)} \le C_\epsilon M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\kappa}{(n+1)(n+2)}+\epsilon} \|f\|_{L^2(\mathbb{R}^n)},
$$

so it remains to verify

$$
M^{-\frac{1}{n+2}} \lambda^{\frac{n}{(n+1)(n+2)}} \le K^{O(1)} M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}, \qquad (4.14)
$$

that is, $M \le K^{O(1)}\gamma^2 R^\kappa$.

However, the second equivalent inequality of (4.14) follows from definition (3.1) of γ, which ensures that $M \le \gamma R^\kappa$ and $\gamma \ge K^{-2\kappa}$. □

4.2.2 The narrow case

In order to prove the narrow case of Theorem 3.1, we have the following lemma, which is essentially contained in Bourgain-Demeter [28]:

Lemma 4.5 Suppose that

(i) B is a narrow K^2-cube in $\mathbb{R}^{n+1}$ that takes $c(B)$ as its center;

(ii) $\mathcal{S}$ denotes the set of K^{-1}-cubes which tile $\mathbb{B}^n$;

(iii) ω_B is a weight function which is essentially a characteristic function on B; more precisely, that

$$
\operatorname{supp} \widehat{\omega_B} \subset B(0, K^{-2}) \quad \text{and} \quad \chi_B(\tilde{x}) \lesssim \omega_B(\tilde{x}) \le \left(1 + \frac{|\tilde{x} - c(B)|}{K^2}\right)^{-1000n}.
$$

Then,

$$
\left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|_{L^p(B)} \le C_\epsilon K^\epsilon \left(\sum_{\tau\in\mathcal{S}} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f_\tau \right\|^2_{L^p(\omega_B)} \right)^{\frac{1}{2}} \quad \forall \ \epsilon > 0.
$$

Next, we prove the narrow case of Theorem 3.1.

Proof The main method we use is the parabolic rescaling and induction on the radius. We prove the narrow case step by step.

First, we consider the wave packet decomposition, which is similar to Theorem 4.1, but with a different scale. We break the physical ball $B^n(0,R)$ into $\frac{R}{K}$-cubes D. From [29], we have that

$$f=\sum_{\tau,D} f_{T_{\tau,D}} \quad \text{and} \quad f_{T_{\tau,D}}=(\hat{f}\chi_\tau)^\vee \chi_D.$$

By computation, we have that $e^{\mathrm{i}t(-\Delta)^\alpha} f_{T_{\tau,D}}$ (whenever restricted to $B^{n+1}(0,R)$) is essentially supported on an $\frac{R}{K}\times\ldots\times\frac{R}{K}\times R$-box, denoted by

$$T_{\tau,D}=\left\{(x,t):(x,t)\in B^{n+1}(0,R) \text{ and } \left|x-c(D)-2t\alpha|c(\tau)|^{2\alpha-2}c(\tau)\right|\le\frac{R}{K} \ \ \& \ \ 0<t<R\right\}.$$

Here $c(\tau)$ and $c(D)$ denote the centers of τ and D, respectively. For a fixed τ, the different tubes $T_{\tau,D}$ tile $B^{n+1}(0,R)$. Next, we write $f=\sum\limits_T f_T$, for convenience. Therefore, by a decoupling theorem, for each narrow K^2-cube B, we have that

$$\left\|e^{\mathrm{i}t(-\Delta)^\alpha}f\right\|_{L^p(B)}\lesssim K^{\epsilon^4}\left(\sum_T\left\|e^{\mathrm{i}t(-\Delta)^\alpha}f_T\right\|^2_{L^p(\omega_B)}\right)^{\frac12}. \tag{4.15}$$

The reason for taking K^{ϵ^4} in (4.15) is that there is a $\frac{1}{K^{2\epsilon}}$ satisfying $\frac{K^{3\epsilon^4}}{K^{2\epsilon}}\ll 1$ at the end of the proof.

Second, we perform a dyadic pigeonholing to get our inductive hypothesis for each f_T. Note that

$$\begin{cases} K=R^\delta=R^{\epsilon^{100}};\\ R_1=\dfrac{R}{K^2}=R^{1-2\delta};\\ K_1=R_1^\delta=R^{\delta-2\delta^2}.\end{cases}$$

Thus, not only tiling the box T by $KK_1^2\times\ldots\times KK_1^2\times K^2K_1^2$-tubes S, but also tiling the box T by $R^{\frac12}\times\ldots\times R^{\frac12}\times KR^{\frac12}$-tubes S' which are running parallel to the long axis of box T, we utilize the parabolic rescaling to reveal that the box T becomes an R_1-cube, and that the tubes S' and S become lattice $R_1^{\frac12}$-cubes and K_1^2-cubes, respectively (See "Seventh", below, for more details).

Third, we classify the tubes S and S' inside each T as follows:

For dyadic numbers η,β_1, let $\mathbb{S}_{T,\eta,\beta_1}=\big\{S: S\subset T,$ each of which has $\sim\eta$ narrow K^2- cubes in Y_{narrow}, and let $\|e^{\mathrm{i}t(-\Delta)^\alpha}f_T\|_{L^p(S)}\sim\beta_1\big\}$.

Fix η,β_1, and for dyadic number λ_1, let $\mathbb{S}_{T,\eta,\beta_1,\lambda_1}=\big\{S: S\in\mathbb{S}_{T,\eta,\beta_1}$. Then the tube $S'\supset S$ contains $\sim\lambda_1$ tubes from $\mathbb{S}_{T,\eta,\beta_1}\big\}$.

For the fixed η,β_1,λ_1, we sort the boxes T. For dyadic numbers β_2,M_1,γ_1, let $\mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}$ denote the collection of boxes T, each of which satisfies that

$$\|f_T\|_{L^2(\mathbb{R}^n)}\sim\beta_2 \quad\text{and}\quad \#\mathbb{S}_{T,\eta,\beta_1,\lambda_1}\sim M_1$$

and

$$\max_{T_r\subset T:r\ge K_1^2}\frac{\#\{S:S\in\mathbb{S}_{T,\eta,\beta_1,\lambda_1} \ \text{and}\ S\subset T_r\}}{r^\kappa}\sim\gamma_1, \tag{4.16}$$

where T_r are $Kr \times \ldots \times Kr \times K^2 r$-tubes in T which are parallel to the long axis of T.

Fourth, let

$$Y_{T,\eta,\beta_1,\lambda_1} = \bigcup_{S \in \mathbb{S}_{T,\eta,\beta_1,\lambda_1}} S.$$

Then, for Y_{narrow}, we can write

$$\begin{aligned} e^{\mathrm{i}t(-\Delta)^\alpha} f = {} & \Sigma_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \Bigg(\sum_{T \in \mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}} e^{\mathrm{i}t(-\Delta)^\alpha} f_T \cdot \chi_{Y_{T,\eta,\beta_1,\lambda_1}} \Bigg) \\ & + O(R^{-1000n}) \|f\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The error term $O(R^{-1000n})\|f\|_{L^2(\mathbb{R}^n)}$ can be neglected.

In particular, on each narrow B, we have

$$e^{\mathrm{i}t(-\Delta)^\alpha} f = \Sigma_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \Bigg(\sum_{\substack{T \in \mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \\ B \subset Y_{T,\eta,\beta_1,\lambda_1}}} e^{\mathrm{i}t(-\Delta)^\alpha} f_T \Bigg). \tag{4.17}$$

Without loss of generality, we assume that

$$\begin{cases} \|f\|_{L^2(\mathbb{R}^n)} = 1; \\ 1 \le \eta \le K^{O(1)}, R^{-10n} \le \beta_1 \le K^{O(1)}, 1 \le \lambda_1 \le R^{O(1)}; \\ R^{-10n} \le \beta_2 \le 1, 1 \le M_1 \le R^{O(1)}, K^{-2n} \le \gamma_1 \le R^{O(1)}. \end{cases}$$

Therefore, there are only $O(\log R)$ significant choices for each dyadic number.

By (4.17), the pigeonholing, and (4.15), we can choose $\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1$ such that

$$\left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|_{L^p(B)} \lesssim (\log R)^6 K^{\epsilon^4} \Bigg(\sum_{\substack{T \in \mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \\ B \subset Y_{T,\eta,\beta_1,\lambda_1}}} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f_T \right\|^2_{L^p(\omega_B)} \Bigg)^{\frac{1}{2}} \tag{4.18}$$

holds for $\gtrsim (\log R)^{-6}$ narrow K^2-cubes B.

Fifth, we fix $\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1$ for the rest of the proof. Let

$$Y_{T,\eta,\beta_1,\lambda_1} = Y_T \quad \text{and} \quad \mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} = \mathbb{B}.$$

Let $Y' \subset Y_{\text{narrow}}$ be a union of narrow K^2-cubes B each of which obeys (4.18)

and

$$\begin{cases} \#\{T : T \in \mathbb{B} \ \text{ and } \ B \subset Y_T\} \sim \nu \quad \text{for some dyadic number } \ 1 \le \nu \le K^{O(1)}; \\ \#\{B : B \subset Y' \ \text{ and } \ B \text{ are } K^2 - \text{cubes}\} \gtrsim (\log R)^{-7} M. \end{cases} \tag{4.19}$$

By our assumption that $\|e^{\mathrm{i}t(-\Delta)^\alpha} f\|_{L^p(B_k)}$ is essentially constant in $k = 1, 2, \ldots, M$, in the narrow case, we have that

$$\left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|^p_{L^p(Y)} \lesssim (\log R)^7 \sum_{B \subset Y'} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|^p_{L^p(B)}. \tag{4.20}$$

For each $B \subset Y'$, it follows from (4.18), Hölder's inequality, and (4.19), that

$$\begin{aligned} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f \right\|^p_{L^p(B)} & \lesssim (\log R)^{6p} K^{\epsilon^4 p} \Bigg(\sum_{T \in \mathbb{B} : B \subset Y_T} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f_T \right\|^2_{L^p(\omega_B)} \Bigg)^{\frac{p}{2}} \\ & \lesssim (\log R)^{6p} K^{\epsilon^4 p} \nu^{\frac{p}{2}-1} \sum_{T \in \mathbb{B} : B \subset Y_T} \left\| e^{\mathrm{i}t(-\Delta)^\alpha} f_T \right\|^p_{L^p(\omega_B)}. \end{aligned} \tag{4.21}$$

By (4.20) and (4.21), we have

$$\begin{aligned}\left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(Y)} &\lesssim (\log R)^{\frac{7}{p}} \left(\sum_{B\subset Y'} \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f\right\|_{L^p(B)}^p\right)^{\frac{1}{p}} \\ &\lesssim (\log R)^{\frac{7}{p}} \left(\sum_{B\subset Y'} (\log R)^{6p} K^{\epsilon^4 p} \nu^{\frac{p}{2}-1} \sum_{T\in\mathbb{B}:B\subset Y_T} \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f_T\right\|_{L^p(\omega_B)}^p\right)^{\frac{1}{p}} \\ &\lesssim (\log R)^{13} K^{\epsilon^4} \nu^{\frac{1}{n+1}} \left(\sum_{B\subset Y'} \sum_{T\in\mathbb{B}:B\subset Y_T} \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f_T\right\|_{L^p(\omega_B)}^p\right)^{\frac{1}{p}} \\ &\lesssim (\log R)^{13} K^{\epsilon^4} \nu^{\frac{1}{n+1}} \left(\sum_{T\in\mathbb{B}} \left\|e^{\mathrm{i}t(-\Delta)^\alpha} f_T\right\|_{L^p(Y_T)}^p\right)^{\frac{1}{p}}. \end{aligned} \tag{4.22}$$

Sixth, regarding each $\|e^{\mathrm{i}t(-\Delta)^\alpha} f_T\|_{L^p(Y_T)}$, we apply parabolic rescaling and induction on the radius. For each K^{-1}-cube $\tau=\tau_T$ in $\mathbb{B}^n$, we write $\xi=\xi_0+K^{-1}\eta\in\tau$, where $\xi_0=c(\tau)$. In a fashion similar to the argument in (4.6), we also consider a collection of the normalized phase functions

$$\mathcal{NPF}(L,c_0)=\left\{\Phi\in C_0^\infty(B^n(0,2)) : \left\|\Phi(\xi)-\frac{|\xi|^2}{2}\right\|_{C^L(\mathbb{B}^n)}\le c_0\right\}.$$

By a similar parabolic rescaling,

$$\begin{cases}\tilde{x}=K^{-1}H^{-t}(x+t\nabla\Phi(\xi_0));\\ \tilde{t}=K^{-2}t,\end{cases}$$

we reach

$$\|e^{\mathrm{i}t\Phi} f_T(x)\|_{L^p(Y_T)}=K^{-\frac{1}{n+1}}|H|^{-\frac{1}{n+1}}\|e^{\mathrm{i}\tilde{t}\Phi_{K^{-1},\xi_0}}g(\tilde{x})\|_{L^p(\tilde{Y})}\sim K^{-\frac{1}{n+1}}\|e^{\mathrm{i}\tilde{t}\Phi_{K^{-1},\xi_0}}g(\tilde{x})\|_{L^p(\tilde{Y})}, \tag{4.23}$$

where

$$\begin{cases}|H|\sim 1 \text{ (since } |\xi|\sim 1);\\ \operatorname{supp}\hat{g}\subset\mathbb{B}^n;\\ \|g\|_{L^2(\mathbb{R}^n)}=\|f_T\|_{L^2(\mathbb{R}^n)}.\end{cases}$$

We also get that $\tilde{Y}$ is the image of Y_T under the new coordinates, and that Φ_{K^{-1},ξ_0} is similar to (4.5).

Seventh, we apply inductive hypothesis (3.2) (replacing $(-\Delta)^\alpha$ with Φ) at scale $R_1=\frac{R}{K^2}$ to $\|e^{\mathrm{i}\tilde{t}(-\Delta)^\alpha}g(\tilde{x})\|_{L^p(\tilde{Y})}$ with $M_1,\gamma_1,\lambda_1,R_1$. Under parabolic rescaling, the relation between the preimage and the image is as follows:

$$\begin{cases}T\ \left(\dfrac{R}{K}\times\ldots\times\dfrac{R}{K}\times R-\text{tube}\right)\longrightarrow\tilde{T}\ \ (R_1-\text{cube});\\ S'\ \left(R^{\frac12}\times\ldots\times R^{\frac12}\times KR^{\frac12}-\text{tube}\right)\longrightarrow\tilde{S}'\ \left(R_1^{\frac12}-\text{cube}\right);\\ S\ \left(KK_1^2\times\ldots\times KK_1^2\times K^2K_1^2-\text{tube}\right)\longrightarrow\tilde{S}\ \left(K_1^2-\text{cube}\right).\end{cases}$$

More precisely, we have that

$$\#\{\tilde{S}:\tilde{S}\subset\tilde{T}\ \ \&\ \ \tilde{S}\subset\tilde{Y}\}\sim M_1,$$

and the K_1^2-cubes $\tilde S$ are organized into $R_1^{\frac12}$-cubes $\tilde S'$ such that

$$\#\{\tilde S:\tilde S\subset \tilde S'\}\sim\lambda_1.$$

Moreover, $\|e^{\mathrm{i}\tilde t(-\Delta)^\alpha}g(\tilde x)\|_{L^p(\tilde S)}$ is dyadically a constant in $S\subset Y_T$. By our choice of γ_1, we have that

$$\max_{\substack{B^{n+1}(x',r)\subset\tilde T\\ x'\in\mathbb{R}^{n+1},r\ge K_1^2}}\frac{\#\{\tilde S:\tilde S\subset B^{n+1}(x',r)\}}{r^\kappa}\sim\gamma_1.$$

Hence, by the inductive hypothesis (3.2) (replacing $(-\Delta)^\alpha$ with Φ) at scale R_1, we have that

$$\|e^{\mathrm{i}\tilde t\Phi_{K^{-1},\xi_0}}g(\tilde x)\|_{L^p(\tilde Y)}\lesssim M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|g\|_{L^2(\mathbb{R}^n)}.$$

By (4.23) and $\|g\|_{L^2(\mathbb{R}^n)}=\|f_T\|_{L^2(\mathbb{R}^n)}$, we get that

$$\begin{aligned}&\|e^{\mathrm{i}t\Phi}f_T(x)\|_{L^p(Y_T)}\\ \lesssim\ & K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f_T\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.24}$$

Since (4.24) also holds whenever one replaces Φ with $(-\Delta)^\alpha$, we get that

$$\begin{aligned}&\|e^{\mathrm{i}t(-\Delta)^\alpha}f_T(x)\|_{L^p(Y_T)}\\ \lesssim\ & K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f_T\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.25}$$

By (4.22) and (4.25), we obtain that

$$\begin{aligned}\|e^{\mathrm{i}t(-\Delta)^\alpha}f\|_{L^p(Y)}\lesssim\ &(\log R)^{13}K^{\epsilon^4}\nu^{\frac{1}{n+1}}\Bigg(\sum_{T\in\mathbb{B}}\Bigg(K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\\ &\times\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f_T\|_{L^2(\mathbb{R}^n)}\Bigg)^p\Bigg)^{\frac1p}\\ \lesssim\ &K^{2\epsilon^4}\nu^{\frac{1}{n+1}}K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\\ &\times\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\left(\sum_{T\in\mathbb{B}}\|f_T\|^p_{L^2(\mathbb{R}^n)}\right)^{\frac1p}\\ \lesssim\ &K^{2\epsilon^4}\left(\frac{\nu}{\#\mathbb{B}}\right)^{\frac{1}{n+1}}K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\\ &\times\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)},\end{aligned}\tag{4.26}$$

where the third inequality follows from the assumption that $\|f_T\|_{L^2(\mathbb{R}^n)}$ is essentially constant in $T\in\mathbb{B}$, and then implies that

$$\left(\sum_{T\in\mathbb{B}}\|f_T\|^p_{L^2(\mathbb{R}^n)}\right)^{\frac1p}\le\left(\frac{1}{\#\mathbb{B}}\right)^{\frac{1}{n+1}}\left(\sum_{T}\|f_T\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac12}\lesssim\left(\frac{1}{\#\mathbb{B}}\right)^{\frac{1}{n+1}}\|f\|_{L^2(\mathbb{R}^n)}.$$

Eighth, we consider the lower bound and the upper bound of

$$\#\{(T,B):T\in\mathbb{B}\ \text{ and }\ B\subset Y_T\cap Y'\}.$$

On the one hand, by the definition of ν in (4.19), there is a lower bound

$$\#\{(T,B): T\in\mathbb{B} \ \text{ and } \ B\subset Y_T\cap Y'\}\gtrsim (\log R)^{-7}M\nu.$$

On the other hand, by our choices of M_1 and η, for each $T\in\mathbb{B}$,

$$\begin{cases}\#\{S: S\subset Y_T\}\sim M_1;\\ \#\{B: B\subset S \ \text{ and } \ B\subset Y_{\text{narrow}}\}\sim\eta.\end{cases}$$

Thus,

$$\#\{(T,B): T\in\mathbb{B} \ \text{ and } \ B\subset Y_T\cap Y'\}\lesssim (\#\mathbb{B})M_1\eta.$$

Therefore, we get

$$\frac{\nu}{\#\mathbb{B}}\lesssim\frac{(\log R)^7M_1\eta}{M}. \tag{4.27}$$

Ninth, we want to obtain the relation between γ and γ_1. By our choices of γ_1, as in (4.16) and η, we get that

$$\begin{aligned}\gamma_1\cdot\eta&\sim\max_{T_r\subset T: r\geq K_1^2}\frac{\#\{S: S\subset Y_T\cap T_r\}}{r^\kappa}\cdot\#\{B: B\subset S\cap Y_{\text{narrow}} \text{ for any fixed } S\subset Y_T\}\\ &\lesssim\max_{T_r\subset T: r\geq K_1^2}\frac{\#\{B: B\subset Y \ \text{ and } \ B\subset T_r\}}{r^\kappa}\leq\frac{K\gamma(Kr)^\kappa}{r^\kappa}=\gamma K^{\kappa+1}.\end{aligned}$$

Hence,

$$\eta\lesssim\frac{\gamma K^{\kappa+1}}{\gamma_1}. \tag{4.28}$$

Tenth, we complete the proof of Theorem 3.1.

On the one hand,

$$\begin{cases}\#\{S: S\subset S' \ \ \& \ \ S\subset Y_T\}\sim\lambda_1;\\ \#\{B: B\subset S \ \ \& \ \ B\subset Y_{\text{narrow}}\}\sim\eta.\end{cases}$$

On the other hand, we can cover S' by $\sim K$, finitely overlapping $R^{\frac12}$-balls, and each $R^{\frac12}$-ball contains $\lesssim\lambda$ many K^2-cubes in Y.

Thus it follows that

$$\lambda_1\lesssim\frac{K\lambda}{\eta}. \tag{4.29}$$

Inserting (4.27), (4.29) and (4.28) into (4.26) gives that

$$\begin{aligned}\|e^{it(-\Delta)^\alpha}f\|_{L^p(Y)}&\lesssim K^{2\epsilon^4}\left(\frac{(\log R)^7M_1\eta}{M}\right)^{\frac{1}{n+1}}K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\\ &\quad\times\gamma_1^{\frac{2}{(n+1)(n+2)}}\left(\frac{K\lambda}{\eta}\right)^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}\\ &\lesssim\frac{K^{3\epsilon^4}}{K^{2\epsilon}}\left(\frac{\eta\gamma_1}{K^{\kappa+1}}\right)^{\frac{2}{(n+1)(n+2)}}M^{-\frac{1}{n+1}}\lambda^{\frac{n}{(n+1)(n+2)}}R^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}\\ &\lesssim\frac{K^{3\epsilon^4}}{K^{2\epsilon}}M^{-\frac{1}{n+1}}\gamma^{\frac{2}{(n+1)(n+2)}}\lambda^{\frac{n}{(n+1)(n+2)}}R^{\frac{\kappa}{(n+1)(n+2)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where the last inequality follows from (4.28). It is not hard to see that $\frac{K^{3\epsilon^4}}{K^{2\epsilon}}\ll 1$, and by induction, we conclude the argument for the narrow case. $\square$

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